THE LAST INTEGRABLE CASE OF KOZLOV-TRESHCHEV BIRKHOFF INTEGRABLE POTENTIALS

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ABSTRACT. We establish the integrability of the last open case in the Kozlov-Treshchev classification of Birkhoff integrable Hamiltonian systems. The technique used is a modification of the so called quadratic Lax pair for D_n Toda lattice combined with a method used by M. Ranada in proving the integrability of the Sklyanin case.

1. INTRODUCTION

In this paper we prove the complete integrability of the Hamiltonian system defined by

$$H=\sum_{i=1}^{n}\frac{1}{2}p_i^2+\sum_{i=1}^{n-1}e^{q_i-q_{i+1}}+e^{q_{n-1}+q_n}+e^{-q_1}+e^{-2q_1}\ . \tag{1}$$

The integrability of this system was conjectured in [16] and in the book of V.V. Kozlov [17]. This system appears first in the classification of Birkhoff integrable systems by Kozlov and Treshchev [16]. The classification involves systems with exponential interraction with sufficient number of integrals, polynomial in the momenta. The classification gives necessary conditions for a system with exponential interraction to be Birkhoff integrable. The integrability (or not) of each system in the list was established case by case using various techniques. The only open case which remains is the case of system (1). For this reason, this last case became sort of famous and we refer to it as the last integrable case of Kozlov-Treshchev potentials. We now give a brief historical review of this area, including previous progress in establishing the integrability of Birkhoff integrable systems.

We begin with the following general definition which involves systems with exponential interaction: Consider a Hamiltonian of the form

$$H=\frac{1}{2}(\mathbf{p},\mathbf{p})+\sum_{i=1}^{N}e^{(\mathbf{v}_i\,,\mathbf{q})}\ , \tag{2}$$

where $\mathbf{q}=(q_1,\ldots,q_n)$, $\mathbf{p}=(p_1,\ldots,p_n)$, $\mathbf{v}_1,\ldots,\mathbf{v}_N$ are vectors in $\mathbf{R}^n$ and $(\ ,\ )$ is the standard inner product in $\mathbf{R}^n$. Following [17] we call the set of vectors $\Delta=\{\mathbf{v}_1,\ldots,\mathbf{v}_N\}$ the spectrum of the system.

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Let M be the $N \times N$ matrix whose elements are

$$M_{ij} = (\mathbf{v}_i, \mathbf{v}_j) \ .$$

Hamilton's equations of motion can be transformed by a generalized Flaschka transformation to a polynomial system of $2N$ differential equations. The transformation is defined as follows:

$$a_i = -e^{(\mathbf{v}_i, \mathbf{q})}, \qquad b_i = (\mathbf{v}_i, \mathbf{p}) \ . \tag{3}$$

We end–up with a system of polynomial differential equations

$$\begin{aligned} \dot{a}_k &= a_k b_k \\ \dot{b}_k &= \sum_{i=1}^{N} M_{ki} a_i \ . \end{aligned} \tag{4}$$

Equations (4) admit the following two integrals

$$F_1 = \sum_{i=1}^{N} \lambda_i b_i \ , \qquad F_2 = \prod_{i=1}^{N} a_i^{\lambda_i} \tag{5}$$

provided that there exist constants λ_i such that $\sum_{i=1}^{N} \lambda_i \mathbf{v}_i = 0$. Such integrals always exist for $N > n$. One can define a canonical bracket on the space of variables (a_i, b_i) by the formula

$$\{b_i, a_j\} = (\mathbf{v}_i, \mathbf{v}_j) a_j$$

and all other brackets equal to zero. The integrals F_1 and F_2 are Casimirs of this bracket.

An interesting special case of (2) occurs when the spectrum is a system of simple roots for a simple Lie algebra $\mathcal{G}$. In this case $N = l = \operatorname{rank} \mathcal{G}$. It is worth mentioning that the case where N, n are arbitrary is an open and unexplored area of research. The main exception is the work of Kozlov and Treshchev [16] where a classification of system (2) is performed under the assumption that the system possesses n polynomial (in the momenta) integrals. We also note the papers by Ranada [22], Annamalai, Tamizhmani [3], Emelyanov [9], Emelyanov and Tsygvintsev [10]. Such systems are called Birkhoff integrable. For each Hamiltonian in (2) we associate a Dynkin type diagram as follows: It is a graph whose vertices correspond to the elements of Δ. Each pair of vertices $\mathbf{v}_i$, $\mathbf{v}_j$ are connected by

$$\frac{4(\mathbf{v}_i, \mathbf{v}_j)^2}{(\mathbf{v}_i, \mathbf{v}_i)(\mathbf{v}_j, \mathbf{v}_j)}$$

edges.

Example 1. The origin of systems of exponential interraction is the classical Toda lattice which corresponds to a Lie algebra of type A_{n-1}. In other words $N = l = n-1$ and we choose Δ to be the set:

$$\mathbf{v}_1 = (1, -1, 0, \ldots, 0), \ldots\ldots\ldots, \mathbf{v}_{n-1} = (0, 0, \ldots, 0, 1, -1) \ .$$

The graph is the usual Dynkin diagram of a Lie algebra of type A_{n-1}. The Hamiltonian becomes:

$$H(q_1, \ldots, q_n,\, p_1, \ldots, p_n) = \sum_{i=1}^{n} \frac{1}{2}\, p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} \ , \tag{6}$$

which is the well-known classical, non–periodic Toda lattice. The interability of this system was established in [11], [12], [14], [18], [19].

As we already mentioned, the Toda lattice was generalized to the case where the spectrum corresponds to a root space of an arbitrary simple Lie group. This generalization is due to Bogoyavlensky [4]. These systems were studied extensively in [15] where the solution of the systems was connected intimately with the representation theory of simple Lie groups. There are also studies by Olshanetsky and Perelomov [20] and Adler, van Moerbeke [2]. The case of D_n-Toda lattice plays an important role in the present paper.

It is more convenient to work, instead with the space of the natural (q, p) variables, with the Flaschka variables (a, b) which are defined by:

$$\begin{array}{rclr} a_i & = & \frac{1}{2} e^{\frac{1}{2}(\mathbf{v}_i, \mathbf{q})} & i = 1, 2, \ldots, N \\ b_i & = & -\frac{1}{2} p_i & i = 1, 2, \ldots, n \ . \end{array} \tag{7}$$

We end–up with a new set of polynomial equations in the variables (a, b). One can write the equations in Lax pair form, see for example [21]. The Lax pair $(L(t), B(t))$ in $\mathcal{G}$ can be described in terms of the root system as follows:

$$L(t) = \sum_{i=1}^{l} b_i(t) h_{\alpha_i} + \sum_{i=1}^{l} a_i(t)(e_{\alpha_i} + e_{-\alpha_i}) \ ,$$

$$B(t) = \sum_{i=1}^{l} a_i(t)(e_{\alpha_i} - e_{-\alpha_i}) \ .$$

As usual h_{α_i} is an element of a fixed Cartan subalgebra and e_{α_i} is a root vector corresponding to the simple root α_i. The Chevalley invariants of $\mathcal{G}$ provide for the constants of motion. In this paper we make use of transformation (7) as opposed to transformation (3).

The first important result in the search for integrable cases of system (2) is due to Adler and van Moerbeke [1]. They considered the special case where the number of elements in the spectrum Δ is $n+1$ (i.e., $N = n+1$). Furthermore, they made the assumption that any n vectors

in the spectrum are independent. Under these conditions a criterion for algebraic integrability is that

$$\frac{2(\mathbf{v}_i,\, \mathbf{v}_j)}{(\mathbf{v}_i,\, \mathbf{v}_i)} \tag{8}$$

should be in the set $\{0, -1, -2, \dots\}$ for all $i \neq j$. The method of proof in [1] is based on the classical method of Kovalevskaya. The classification obtained corresponds to the simple roots of graded Kac–Moody algebras. The associated systems are the periodic Toda lattices of Bogoyavlensky [4]. The complete integrability of these systems using Lax pairs with a spectral parameter was already established in [2].

The next development in the study of system (2) is the work of Kozlov and Treshchev on Birkhoff integrable systems. A system of the form (2) is called Birkhoff integrable if it has n integrals, polynomial in the momenta with coefficients of the form

$$\sum \lambda_j e^{<\mathbf{u}_j,\, \mathbf{q}>} \ , \quad \lambda_j \in \mathbf{R}, \quad \mathbf{u}_j \in \mathbf{R}^n \ ,$$

whose leading homogeneous forms are almost everywhere independent. We remark that in the definition given in the book of Kozlov [17] there is no assumption on involutivity of the integrals. In [16] it is proved that the polynomial integrals are in involution. For this reason, we will not deal with the involution of the integrals; it follows from Lemma 5, p. 567 of [16]. The terminology has its origin in the work of Birkhoff who studied the conditions for the existence of linear and quadratic integrals of general Hamiltonians in two degrees of freedom. A vector in Δ is called maximal if it has the greatest possible length among all the vectors in the spectrum having the same direction. Kozlov and Treshchev proved the following theorem:

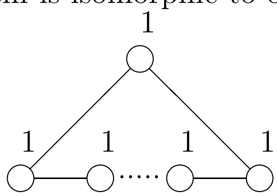
Theorem 1. *Assume that the Hamiltonian (2) is Birkhoff integrable. Let $\mathbf{v}_i$ be a maximal vector in Δ and assume that the vector $\mathbf{v}_j \in \Delta$ is linearly independent of $\mathbf{v}_i$. Then*

$$\frac{2(\mathbf{v}_i,\, \mathbf{v}_j)}{(\mathbf{v}_i,\, \mathbf{v}_i)}$$

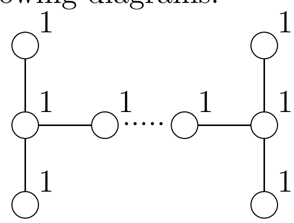
lies in the set $\{0, -1, -2, \dots\}$.

Note that the condition of the theorem is exactly the same as condition (8) of Adler and van Moerbeke. Of course theorem 1 is more general since there is no restriction on the integer N (the number of summands in the potential of (2)). It turns out, however, that N cannot be much bigger than n. In fact, it follows from the classification that $N \leq n + 3$. A system of the form (2) is called complete if there exist no vector $\mathbf{v}$ such that the set $\Delta \cup \{\mathbf{v}\}$ satisfies the assumptions of Theorem 1. In [16] there is a complete classification of all possible Birkhoff integrable systems based on Theorem 1. The Dynkin type

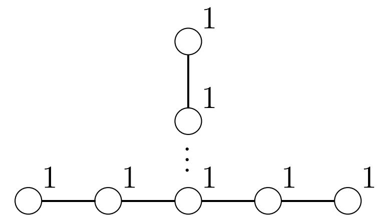
diagram of a complete, irreducible, Birkhoff integrable Hamiltonian system is isomorphic to one of the following diagrams:



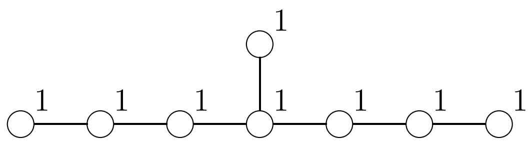
(a)



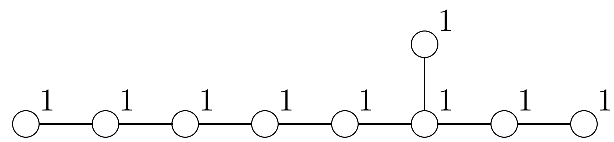
(b)



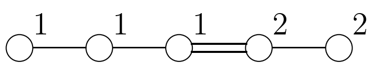
(c)



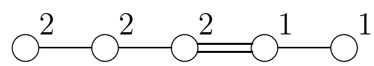
(d)



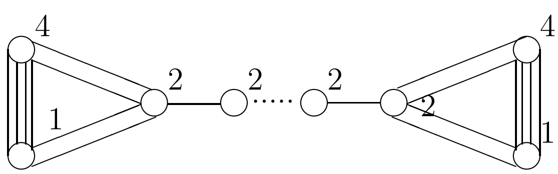
(e)



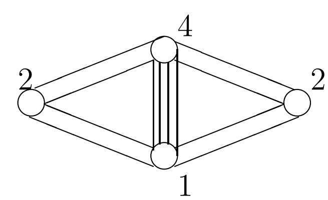
(f)



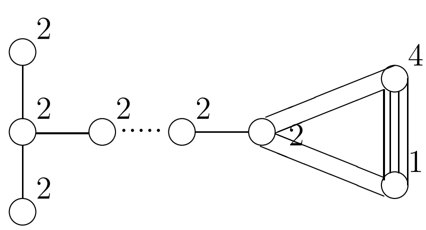
(g)



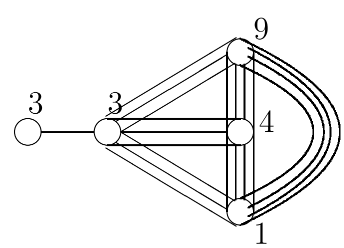
(h)



(i)



(j)



(k)

Remark 1 In the list of diagrams we have omitted some cases that occur as sub–graphs of diagrams (a)–(k) (by truncating one or more vertices).

Remark 2 The Dynkin type diagram determines only the angles between pairs of vectors in Δ. In order to reconstruct the ratios of lengths of vectors in Δ we assign to the ith vertex a coefficient proportional to the square of the length of $\mathbf{v}_i$. This explains the numbers appearing on the vertices of the diagrams.

We have to stress that this classification gives only necessary conditions for a system of type (2) to be Birkhoff integrable. The integrability for each system in the list should be established case by case. As we already mentioned, the integrability of systems (a)–(g) was established in [2], [4]. The solution of these generalized periodic Toda lattices (associated with affine Lie algebras) was obtained by Goodman and Wallach in [13]. The graph (i) corresponds to a Hamiltonian system in two degrees of freedom with potential

$$e^{q_1} + e^{q_2} + e^{-q_1-q_2} + e^{-\left(\frac{q_1+q_2}{2}\right)} .$$

The additional integral can be found in [16].

Sklyanin [23] pointed out another integrable generalization of the Toda lattice:

(9)
$$H(q_1, \ldots, q_n, p_1, \ldots, p_n) = \sum_{i=1}^{n} \frac{1}{2} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} + \alpha_1 e^{q_1} + \beta_1 e^{2q_1} + \alpha_n e^{-q_n} + \beta_n e^{-2q_n} .$$

He obtained this system by means of the quantum inverse scattering R-matrix method. This corresponds to system (h) with associated Hamiltonian (9). The case $n = 2$ corresponds to the potential

$$V = e^{q_1 - q_2} + c_1 e^{2q_2} + c_2 e^{q_2} + c_3^{-q_1} + c_4 e^{-2q_1} .$$

Annamalai and Tamizhmani [3] demonstrated the integrability of this particular case by using Noether's theorem. The second integral is of fourth degree in the momenta. A special case of (9) was considered in [8] using the generalization of KM (Volterra) system due to Bogoyavlensky. The case $n = 3$ (as well as the general case) is treated in Ranada [22]. Ranada proved integrability by using a Lax pair approach. The additional integrals are of degree 4 and 6. Ranada's approach will be used in proving the integrability of system (1). It is generally believed that system (k) in non-integrable. In fact the arguments in [16] support the non-integrability of system (k). This leaves only system (j) which is treated in the present paper.

The only essential progress to our knowledge in proving the integrability of system (1) are the two papers [9] and [10]. The case $n = 4$ was studied in [10]. Based on calculations of the Kovalevskaya exponents it was conjectured that the degrees of homogeneity of the additional first integrals are $4, 6, 8$. Our results confirm that prediction. We reproduce here a small part of the table from [10] which contains the Kovalevskaya exponents corresponding to the various values of the indicial locus c:

Vector $\mathbf{c}$	KE exponents
(0,6,0,10,6,0)	-5,-3,-2,-1, 1,1,1,2,4,6,8,8
(3,4,3,3,0,0)	-3,-2,-1, 1, 1,1,2,3,4,5,6,8
(7,12,0,15,0,16)	-7,-5,-3,-1, 1,1,1,2,4,6,8,14
(8,14,0,18,10,0)	-7,-5,-3,-2,-1,1,1,2,4,6,8,16
(1,0,1,4,3,0)	-3,-2,-1,-1,-1,1,1,2,2,2,4,8
(3,4,3,0,1,0)	-3,-2,-2,-1,-1,1,1,2,2,3,4,8
(7,14,0,18,10,0)	-7,-5,-3,-2,-1,1,1,2,4,6,8,16

In a brief communication by Emelyanov [9] it was reported (without proof) that the system (1) was integrable for $n = 4$. In fact a few leading terms of the integrals were displayed but our leading terms of the integrals do not match exactly those results. However, it is possible that the integrals calculated by the author of [9] are combinations of the integrals of the present paper. To quote from [9]: "A study of Kovalewski indices shows that it is possible to assume for $n = 5$ the degrees of additional integrals are $4, 6, 8, 10$ etc. If integrability is established for an arbitrary n, then one can state that only systems whose Dynkin schemes are isomorphic to those of Fig. 1 except for scheme (k) are Birkhoff integrable".

2. D_n Toda systems

Since the Lax pair of system (1) uses in an essential way the Lax pair of the D_n Toda lattice we include a review of this Bogoyavlensky-Toda system following [5], [6], [7].

The Hamiltonian for the D_n Toda lattice is

$$H = \frac{1}{2}\sum_{1}^{n} p_j^2 + e^{q_1 - q_2} + \cdots + e^{q_{n-1}-q_n} + e^{q_{n-1}+q_n} \qquad n \geq 4 \ . \tag{10}$$

We make a Flaschka-type transformation, $F : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ defined by

$$F : (q_1, \ldots, q_n, p_1, \ldots, p_n) \to (a_1, \ldots, a_n, b_1, \ldots, b_n) \ ,$$

with

$$a_i = \frac{1}{2}\, e^{\frac{1}{2}(q_i - q_{i+1})} \ , \qquad i = 1, 2, \ldots, n-1, \qquad a_n = \frac{1}{2} e^{\frac{1}{2}(q_{n-1}+q_n)} \ , \tag{11}$$

$$b_i = -\frac{1}{2} p_i, \qquad i = 1, 2, \ldots, n \ .$$

Then

$$
\begin{array}{lcll}
\dot{a}_i & = & a_i\,(b_{i+1}-b_i) & i=1,2,\ldots,n-1 \\
\dot{a}_n & = & -a_n(b_{n-1}+b_n) & \\
\dot{b}_i & = & 2\,(a_i^2-a_{i-1}^2) & i=1,2,\ldots,n-2 \text{ and } i=n \\
\dot{b}_{n-1} & = & 2(a_n^2+a_{n-1}^2-a_{n-2}^2)\ . &
\end{array}
\tag{12}
$$

These equations can be written as a Lax pair $\dot{L}=[B,L]$, where L is the symmetric matrix

$$
\begin{pmatrix}
b_1 & a_1 & & & & & & & \\
a_1 & \ddots & \ddots & & & & & & \\
 & \ddots & \ddots & a_{n-1} & -a_n & 0 & & & \\
 & & a_{n-1} & b_n & 0 & a_n & & & \\
 & & -a_n & 0 & -b_n & -a_{n-1} & & & \\
 & & 0 & a_n & -a_{n-1} & \ddots & \ddots & & \\
 & & & & & \ddots & \ddots & -a_1 \\
 & & & & & & -a_1 & -b_1
\end{pmatrix},
\tag{13}
$$

and B is the skew-symmetric part of L (In the decomposition, lower Borel plus skew-symmetric).

The mapping $F:\mathbf{R}^{2n}\to\mathbf{R}^{2n}$, $(q_i,p_i)\to(a_i,b_i)$, defined by (11), transforms the standard symplectic bracket into another symplectic bracket π_1 given (up to a constant multiple) by

$$
\begin{array}{lcll}
\{a_i,b_i\} & = & -\frac{1}{2}a_i & i=1,2,\ldots,n \\
\{a_i,b_{i+1}\} & = & \frac{1}{2}a_i & i=1,2,\ldots,n-1 \\
\{a_n,b_{n-1}\} & = & -\frac{1}{2}a_n. &
\end{array}
\tag{14}
$$

We obtain a hierarchy of invariant polynomials, which we denote by

$$H_2,\ H_4,\ \ldots,\ H_{2n},\ldots$$

defined by $H_{2i}=\frac{1}{2i}\,\mathrm{Tr}\ L^{2i}$. The degrees of the first $n-1$ (independent) polynomials are $2,4,\ldots,2n-2$. We also define

$$P_n=\sqrt{\det L}\ .$$

The degree of P_n is n. The set $\{H_2,\ H_4,\ \ldots,\ H_{2n-2},P_n\}$ corresponds to the Chevalley invariants for a Lie group of type D_n.

Taking

$$H_2=\frac{1}{2}\mathrm{Tr}\ L^2=\sum_{i=1}^{n}b_i^2+2\sum_{i=1}^{n}a_i^2$$

as the Hamiltonian we have that

$$\pi_1\nabla H_2$$

gives precisely equations (12).

3. QUADRATIC LAX PAIRS

We employ the method used by Ranada in [22]. The key idea is the following: If (L, B) is a Lax pair, so is (L^2, B). This follows easily from

$$\frac{d}{dt}L^2 = [B, L]L + L[B, L] = (BL - LB)L + L(BL - LB) = BL^2 - L^2B = [B, L^2] .$$

In the case of D_4 Toda lattice, L^2 is the matrix

$$\begin{pmatrix} a_1^2+b_1^2 & a_1(b_1+b_2) & a_1a_2 & 0 & 0 & 0 & 0 & 0 \\ a_1(b_1+b_2) & a_1^2+a_2^2+b_2^2 & a_2(b_2+b_3) & a_2a_3 & -a_2a_4 & 0 & 0 & 0 \\ a_1a_2 & a_2(b_2+b_3) & a_2^2+a_3^2+a_4^2+b_3^2 & a_3(b_3+b_4) & a_4(b_4-b_3) & 2a_3a_4 & 0 & 0 \\ 0 & a_2a_3 & a_3(b_3+b_4) & a_3^2+a_4^2+b_4^2 & -2a_3a_4 & a_4(b_4-b_3) & -a_2a_4 & 0 \\ 0 & -a_2a_4 & a_4(b_4-b_3) & -2a_3a_4 & a_3^2+a_4^2+b_4^2 & a_3(b_3+b_4) & a_2a_3 & 0 \\ 0 & 0 & 2a_3a_4 & a_4(b_4-b_3) & a_3(b_3+b_4) & a_2^2+a_3^2+a_4^2+b_3^2 & a_2(b_2+b_3) & a_1a_2 \\ 0 & 0 & 0 & -a_2a_4 & a_2a_3 & a_2(b_2+b_3) & a_1^2+a_2^2+b_2^2 & a_1(b_1+b_2) \\ 0 & 0 & 0 & 0 & 0 & a_1a_2 & a_1(b_1+b_2) & a_1^2+b_1^2 \end{pmatrix} . \tag{15}$$

and

$$B = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_1 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_2 & 0 & a_3 & -a_4 & 0 & 0 & 0 \\ 0 & 0 & -a_3 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & a_4 & 0 & 0 & -a_3 & 0 & 0 \\ 0 & 0 & 0 & -a_4 & a_3 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2 & 0 & -a_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 \end{pmatrix} . \tag{16}$$

Note that the Lax equations $\dot{L}^2 = [B, L^2]$ gives the same equations (12) plus some consistency conditions.

4. THE CASE $n = 4$.

We now consider the special case of (1) for $n = 4$. The general case follows easily from this special case. Let

$$H = \sum_{i=1}^{4} \frac{1}{2}\,p_i^2 + \sum_{i=1}^{3} e^{q_i - q_{i+1}} + e^{q_3+q_4} + e^{-q_1} + e^{-2q_1} \ . \tag{17}$$

We make a Flaschka-type transformation, $F : \mathbf{R}^8 \to \mathbf{R}^9$ defined by

$$F : (q_1, \ldots, q_4, p_1, \ldots, p_4) \to (a_1, \ldots, a_5, b_1, \ldots, b_4) \ ,$$

with

$$a_i = \frac{1}{2}\,e^{\frac{1}{2}(q_i - q_{i+1})} \ , \qquad i = 1, 2, 3, \quad a_4 = \frac{1}{2}e^{\frac{1}{2}(q_3+q_4)} \ , \quad a_5 = \frac{1}{\sqrt{2}}e^{-\frac{1}{2}q_1} \ , \tag{18}$$

$$b_i = -\frac{1}{2}p_i, \qquad i = 1, 2, 3, 4 \ .$$

We obtain the following equations of motion:

$$
\begin{array}{lrcll}
 & \dot{a}_i & = & a_i\,(b_{i+1}-b_i) & i=1,2,3 \\
 & \dot{a}_4 & = & -a_4(b_3+b_4) & \\
 & \dot{a}_5 & = & a_5 b_1 & \\
(19) & \dot{b}_1 & = & 2a_1^2-a_5^2-4a_5^4 & \\
 & \dot{b}_i & = & 2\,(a_i^2-a_{i-1}^2) & i=2 \text{ and } i=4 \\
 & \dot{b}_3 & = & 2(a_4^2+a_3^2-a_2^2)\ . &
\end{array}
$$

Note that the Hamiltonian in the new variables takes the form

$$
H=\sum_{i=1}^{4} b_i^2+2\sum_{i=1}^{4} a_i^2+a_5^2+2a_5^4\ . \tag{20}
$$

The image of the symplectic bracket is now the following extension of bracket (14) in the new phase space $(a_1,\ldots,a_5,b_1,\ldots,b_4)$.

$$
\begin{array}{lrcll}
 & \{a_i,b_i\} & = & -\frac{1}{2}a_i & i=1,2,\ldots,4 \\
 & \{a_5,b_1\} & = & \frac{1}{2}a_5 & \\
(21) & \{a_i,b_{i+1}\} & = & \frac{1}{2}a_i & i=1,2,\ldots,3 \\
 & \{a_4,b_3\} & = & -\frac{1}{2}a_4. &
\end{array}
$$

We denote this bracket by w_1. Using the Hamiltonian (20) and the above bracket w_1 gives equations (19) as is easily checked.

We now present a Lax pair for equations (19). The method of obtaining this Lax pair is based in the ideas of [22] and we omit the details.

The Lax pair has the form $\dot{A}=[C,A]$ where the matrices C and A are perturbations of the matrices B and L^2 respectively. More precisely, A is the same as L^2 except that

$$
A_{11}=a_1^2+b_1^2+a_5^2+2a_5^4=A_{88}\ ,
$$

$$
A_{12}=a_1(b_1+b_2+\sqrt{2}ia_5^2)=A_{87}, \qquad i^2=-1
$$

$$
A_{21}=a_1(b_1+b_2-\sqrt{2}ia_5^2)=A_{78}\ .
$$

On the other hand C differs from B only at two diagonal positions. It is the same as B except that

$$
C_{11}=\sqrt{2}ia_5^2=C_{88}\ .
$$

It is a simple calculation to show that the equations $\dot{A}=[C,A]$ are a matrix form of equations (19).

Define $h_{2i}=\frac{1}{2}\ \mathrm{Tr}\ A^i \quad i=1,2,3,4.$

$$
h_2=\frac{1}{2}\mathrm{Tr}\ A=H=\sum_{i=1}^{4} b_i^2+2\sum_{i=1}^{4} a_i^2+a_5^2+2a_5^4\ .
$$

Let

$$\begin{array}{rcl} t_1 &=& a_1^2 + \frac{1}{2}a_5^2 + a_5^4 \\ t_2 &=& a_1^2 + a_2^2 \\ t_3 &=& a_2^2 + a_3^2 + a_4^2 \\ t_4 &=& a_3^2 + a_4^2 \ . \end{array}$$

Then $h_4 = \frac{1}{2}\text{Tr } A^2$ is given by

$$h_4 = b_1^4 + b_2^4 + b_3^4 + b_4^4 + \sum_{i=1}^{4} 4t_i b_i^2 +$$

$$4a_1^2 b_1 b_2 + 4a_2^2 b_2 b_3 + 4(a_3^2 - a_4^2) b_3 b_4 + s(a_1, a_2, a_3, a_4, a_5) \ ,$$

where

$$s = 4a_1^2a_2^2 + 4a_2^2a_4^2 + 4a_2^2a_3^2 + 12a_3^2a_4^2 + 8a_1^2a_5^4 + 2a_1^2a_5^2 + 2a_3^4 + 2a_4^4 + 2a_1^4 + 2a_2^4 + a_5^4 + 4a_5^6 + 4a_5^8 \ .$$

Similarly h_6 has the form $\sum_{i=1}^4 b_i^6 + \sum_{i=1}^4 12t_i b_i^4 +$ other terms even in the momenta.

Finally h_8 has the form $\sum_{i=1}^4 b_i^8 + \sum_{i=1}^4 8t_i b_i^6 +$ other terms even in the momenta.

The set of functions $\{h_2, h_4, h_6, h_8\}$ provide a maximal set of independent integrals in involution. The fact that the leading terms of the integrals are functions of the momenta only, agrees with the general facts proved in [16].

5. The general case

The procedure for the general case is very similar to the case $n = 4$. There is no interaction between q_1 and the newly introduced variables, therefore the procedure is quite identical.

Consider the Hamiltonian

$$H = \sum_{i=1}^{n} \frac{1}{2} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} + e^{q_{n-1}+q_n} + e^{-q_1} + e^{-2q_1} \ .$$

We make a Flaschka-type transformation, $F : \mathbf{R}^n \to \mathbf{R}^{n+1}$ defined by

$$F : (q_1, \ldots, q_n, p_1, \ldots, p_n) \to (a_1, \ldots, a_{n+1}, b_1, \ldots, b_n) \ ,$$

with

(22)
$$a_i = \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})} \ , \qquad i = 1, 2, \ldots, n-1, \qquad a_n = \frac{1}{2} e^{\frac{1}{2}(q_{n-1}+q_n)} \ , \qquad a_{n+1} = \frac{1}{\sqrt{2}} e^{-\frac{1}{2}q_1} \ ,$$

$$b_i = -\frac{1}{2} p_i, \qquad i = 1, 2, \ldots n \ .$$

We obtain the following equations of motion:

$$
\begin{array}{llll}
 & \dot{a}_i & = & a_i\,(b_{i+1}-b_i) \qquad\qquad i=1,2,\ldots,n-1 \\
 & \dot{a}_n & = & -a_n(b_{n-1}+b_n) \\
 & \dot{a}_{n+1} & = & a_{n+1}b_1 \\
(23) & \dot{b}_1 & = & 2a_1^2-a_{n+1}^2-4a_{n+1}^2 \\
 & \dot{b}_i & = & 2\,(a_i^2-a_{i-1}^2) \qquad\qquad i=2,3,\ldots,n-2 \text{ and } i=n \\
 & \dot{b}_{n-1} & = & 2(a_n^2+a_{n-1}^2-a_{n-2}^2)\ .
\end{array}
$$

Note that the Hamiltonian in the new variables takes the form

$$
H=\sum_{i=1}^{n} b_i^2+2\sum_{i=1}^{n} a_i^2+a_{n+1}^2+2a_{n+1}^4\ . \tag{24}
$$

The image of the symplectic bracket is now the following extension of bracket (14) in the new phase space $(a_1,\ldots,a_{n+1},b_1,\ldots,b_n)$.

$$
\begin{array}{lcll}
\{a_i,b_i\} & = & -\frac{1}{2}a_i & i=1,2,\ldots,n \\
\{a_{n+1},b_1\} & = & \frac{1}{2}a_{n+1} & \\
\{a_i,b_{i+1}\} & = & \frac{1}{2}a_i & i=1,2,\ldots,n-1 \\
\{a_n,b_{n-1}\} & = & -\frac{1}{2}a_n. &
\end{array}
$$

We denote this bracket by w_1. Using the Hamiltonian (24) and the above bracket w_1 gives equations (23) as is easily checked.

We now obtain a Lax pair for equations (23).

The Lax pair has the form $\dot{A}=[C,A]$ where the matrices C and A are perturbations of the matrices B and L^2 respectively. More precisely, A is the same as L^2 except that

$$
A_{11}=a_1^2+b_1^2+a_{n+1}^2+2a_{n+1}^4=A_{nn}\ ,
$$

$$
A_{12}=a_1(b_1+b_2+\sqrt{2}ia_{n+1}^2)=A_{(n+1)n}, \qquad i^2=-1,
$$

$$
A_{21}=a_1(b_1+b_2-\sqrt{2}ia_{n+1}^2)=A_{n(n+1)}\ .
$$

On the other hand C differs from B only at two diagonal positions. It is the same as B except that

$$
C_{11}=\sqrt{2}ia_{n+1}^2=C_{nn}\ .
$$

It is a simple calculation to show that the equations $\dot{A}=[C,A]$ are a matrix form of equations (23).

Define $h_{2i}=\frac{1}{2}\ \mathrm{Tr}\ A^i \quad i=1,2,\ldots,n.$

e.g.,

$$
h_2=\frac{1}{2}\mathrm{Tr}\ A=H=\sum_{i=1}^{n} b_i^2+2\sum_{i=1}^{n} a_i^2+a_{n+1}^2+2a_{n+1}^4\ ,
$$

and h_{2i} is a homogenous polynomial in the b_i of degree $2i$.

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